

# Quantum anharmonic oscillator and quasi-exactly solvable Bose systems

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We extend the notion of quasi-exactly solvable (QES) models from potential ones and differential equations to Bose systems. We obtain conditions under which algebraization of the part of the spectrum occurs. In some particular cases simple exact expressions for several energy levels of an anharmonic Bose oscillator are obtained explicitly. The corresponding results do not exploit perturbation theory and include strong coupling regime. A generic Hamiltonian under discussion cannot, in contrast to QES potential models, be expressed as a polynomial in generators of  $sl_2$  algebra. The suggested approach is extendable to many-particle Bose systems with interaction.

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## I. INTRODUCTION

An anharmonic oscillator represents one of the "eternal" problems and models of theoretical physics. It serves as a basis for checking different approximate methods in quantum mechanics, the simplified counterpart of field-theoretical models, etc. Apart from this, it is of interest on its own since the real world certainly deviates from idealized picture of harmonic oscillators due to interaction between them and self-interaction. In so doing, the notion "anharmonic oscillator" is mainly applied at least to two different entities. First, it

refers to some potential power models in which the potential contains terms with the higher degrees with respect to coordinate. (The literature on this subject is so vast, that it is even rather difficult to indicate some concrete references - let us mention here only the reviews [1], [2], the book [3] and references therein). Second, it is related to quantum Bose models with interaction or self-interaction. In both cases the Schrödinger equation cannot be solved exactly. However, for the first case it was realized that, in spite of impossibility to find the whole energy spectrum exactly, in some particular cases (for instance, sextic oscillator with special relationship between coefficients [4]) one can find the *part* of the spectrum (more precisely, algebraization of the part of the spectrum occurs). Such a system is the example of so-called quasi-exactly solvable (QES) [5] ones which includes a rather vast class of potentials and have direct physical meaning, first of all related to properties of magnetic systems [6].

The aim of the present paper is to extend the notion of QES systems to Bose ones and apply QES approach to anharmonic Bose oscillators. Strange as it may seem, the approach to Bose oscillators in the spirit of QES models was, to the best of our knowledge, absent in literature before in spite of the developed apparatus relating realization of Lie algebras in Fock space and properties of differential equations [7]. Meanwhile, the QES approach to Bose systems deserves treatment on its own due to an obviously wide area of physical applications.

In the coordinate-momentum representation

$$a \rightarrow \frac{d}{dx}, a^+ \rightarrow x, \quad (1)$$

Bose Hamiltonian, polynomial with respect to  $a, a^+$  becomes a differential operator and any possible invariant subspace is spanned on the polynomial basis. For the systems of such a kind there exists the Turbiner's theorem that states that the most general QES can be expressed in terms of generators of  $sl_2$  algebra

$$J^+ = a^{+2}a - Na^+, J^- = a, J^0 = a^+a - \frac{N}{2}. \quad (2)$$

(the relations (2) are known in the magnetism theory as the Dayson-Maleev representation [8]). Nevertheless, the problem of finding QES Bose Hamiltonians cannot be exhausted by a simple reference to this theorem since, as we will below, for the typical case under discussion, the conditions of validity of this theorem are not fulfilled, so Hamiltonians (except some special cases) cannot be expressed in terms of  $sl_2$  generators at all. This is the point in which Bose QES systems qualitatively differ from potential QES models whose Hamiltonians are built with the help of  $sl_2$  generators, realized as differential operators [5].

Apart from this, even in the cases when Turbiner's theorem does apply to Bose Hamiltonians, it is much more convenient to formulate the conditions of QES - solvability in terms of coefficients of an original Bose Hamiltonian directly without resorting to operators  $J^i$  at the intermediate stage.

## II. BASIC FORMULAS

Consider Hamiltonian

$$H = H_0 + V, \quad H_0 = \sum_{p=1}^{p_0} \varepsilon_p (a^+ a)^p, \quad V = \sum_{s=0}^{s_0} A_s [(a^+ a)^s a^2 + (a^+)^2 (a^+ a)^s]. \quad (3)$$

Throughout the paper we assume that all coefficients of Hamiltonian are real. For Hamiltonian (3) to have a well-defined ground state, one should take  $p_0 > s_0 + 2$  independently of the relations between coefficients or  $p_0 = s_0 + 2$  provided  $\varepsilon_{p_0} \geq 2A_{s_0}$ . In the  $x$ -representation (1)

we obtain

$$H_x = \sum_{p=1}^{p_0} \varepsilon_p \left(x \frac{d}{dx}\right)^p + \sum_{s=0}^{s_0} A_s \left[\left(x \frac{d}{dx}\right)^s \frac{d^2}{dx^2} + x^2 \left(x \frac{d}{dx}\right)^s\right]. \quad (4)$$

We are interested in the solutions of Schrödinger equation of the type  $|\psi\rangle = \sum_{n=0}^N b_n |n\rangle$ , where  $|n\rangle$  is the state with  $n$  particles:  $a^+ a |n\rangle = n |n\rangle$ . For Hamiltonian (4) subspaces with even and odd are not mixed. Therefore, it makes sense to consider them separately. In  $x$  representation (1) the wave function of even states  $\Phi = \sum_{l=0} a_l \Phi_l$ ,  $\Phi_l \equiv x^{2l}$ .

It follows from (4) that

$$\begin{aligned}
H_x \Phi_l &= \alpha_l \Phi_{l+1} + \beta_l \Phi_{l-1} + \gamma_l \Phi_l, \quad l = 0, 1, \dots, L, \\
\alpha_l &= \sum_{s=0}^{s_0} A_s (2l)^s, \\
\beta_l &= \sum_{s=0}^{s_0} A_s 2l(2l-1)(2l-2)^s, \\
\gamma_l &= \sum_{p=1}^{p_0} \varepsilon_p (2l)^p.
\end{aligned} \tag{5}$$

We are interested in the possibility of the existence of the invariant basic  $F_{2L} = \{1, x^2, x^4, \dots, x^{2L}\}$ . The condition of cut off at  $l = L$  reads:  $\alpha_L = 0$ .

For odd states the invariant basic  $F_{2M+1} = \{x, x^3, \dots, x^{2M+1}\}$ ,  $\tilde{\Phi}_m = x^{2m+1}$  and

$$\begin{aligned}
H_x \tilde{\Phi}_m &= \tilde{\alpha}_m \tilde{\Phi}_{m+1} + \tilde{\beta}_m \tilde{\Phi}_{m-1} + \tilde{\gamma}_m \tilde{\Phi}_m, \quad m = 0, 1, \dots, M, \\
\tilde{\alpha}_m &= \sum_{s=0}^{s=s_0} A_s (2m+1)^s, \\
\tilde{\beta}_m &= \sum_{s=0}^{s=s_0} A_s (2m+1) 2m(2m-1)^s, \\
\tilde{\gamma}_m &= \sum_{p=1}^{p=p_0} \varepsilon_p (2m+1)^p.
\end{aligned} \tag{6}$$

The subspace with  $m \leq M$  is invariant with respect to the action of  $H$  provided  $\tilde{\alpha}_M = 0$ .

The procedure described above is, in fact, nothing else than the Bose version of quasi-exactly solvable (QES) models, applied to an anharmonic oscillator. Now we would like to point out why for the case under consideration Turbiner's theorem, in general, does not hold, so our formulas cannot be considered as particular cases of its realization. The point is that Turbiner's theorem implies that the space of *all* polynomials of a given degree is invariant with respect to  $J^i$ :  $F_N = \{1, x, x^2, \dots, x^N\}$ . Meanwhile, in our case, only subset  $F_{2N}$  (for even states) or  $F_{2M+1}$  (for odd ones) is invariant, whereas the set  $F_N$  is not. Only in particular cases, when both conditions  $\alpha_L = 0$  (for even states) and  $\tilde{\alpha}_M = 0$  (for odd states) are satisfied simultaneously, Hamiltonian does become an algebraic combination of  $J^i$ .

In contrast to [11], where differential equations were the object of research, in our paper the coordinate-momentum representation (1), in which the operator  $a$  becomes differential,

is used as an useful device at an intermediate stage only. In principle, one could rely directly on the known formulas of the action of operators  $a$ ,  $a^+$  on a states with a definite number of particles without resorting to the representation (1).

### III. EXAMPLES

Consider Hamiltonian whose off-diagonal part reads

$$V = A_0(a^2 + a^{+2}) + A_1[(a^+a)a^2 + a^{+2}(a^+a)] + A_2[(a^+a)^2a^2 + a^{+2}(a^+a)^2], \quad s_0 = 2. \quad (7)$$

Now

$$\alpha_l = A_0 + 2lA_1 + (2l)^2A_2, \quad (8)$$

$$\beta_l = 2l(2l-1)[A_0 + (2l-2)A_1 + (2l-2)^2A_2].$$

$$\tilde{\alpha}_m = A_0 + (2m+1)A_1 + (2m+1)^2A_2, \quad (9)$$

$$\tilde{\beta}_m = (2m+1)2m[A_0 + (2m-1)A_1 + (2m-1)^2A_2].$$

First, consider even states. In the simplest nontrivial particular case the invariant subspace is two-dimensional,  $L = 1$ . Then

$$\alpha_1 = A_0 + 2A_1 + 4A_2 = 0, \quad (10)$$

$\Phi = a_0\Phi_0 + a_1\Phi_1$  and it follows from the Schrödinger equation  $H\Phi = E\Phi$  that  $-Ea_0 + \beta_1a_1 = 0$ ,  $\alpha_0a_0 + (\gamma_1 - E)a_1 = 0$ . Taking also into account (10), we obtain:

$$E = \frac{\gamma_1}{2} \pm \sqrt{\frac{\gamma_1^2}{4} + 8(A_1 + 2A_2)^2} \quad (11)$$

In a similar way, one gets for the three-dimensional subspace ( $L = 2$ ):

$$E^3 - (\gamma_1 + \gamma_2)E^2 + [\gamma_1\gamma_2 - 16(5A_1^2 + 52A_1A_2 + 140A_2^2)]E + 32\gamma_2(A_1 + 4A_2)^2 = 0. \quad (12)$$

If  $A_1 = A_2 = 0$ , the low-lying energy levels of an harmonic oscillator are reproduced from (11), (12). The equation (12) can be solved exactly in the particular case  $A_1 = -4A_2$ :

$E = 0$ ,  $\frac{\gamma_1 + \gamma_2}{2} \pm \sqrt{\frac{(\gamma_1 - \gamma_2)^2}{4} + 192A_2^2}$ . For odd states in the simplest nontrivial case  $M = 1$  we have

$$\tilde{\alpha}_1 \equiv A_0 + 3A_1 + 9A_2 = 0, \quad (13)$$

$$\begin{aligned} \tilde{\alpha}_0 &= A_0 + A_1 + A_2, \quad \tilde{\beta}_1 = 6\tilde{\alpha}_0, \\ E &= \frac{\tilde{\gamma}_0 + \tilde{\gamma}_1}{2} \pm \sqrt{\frac{(\tilde{\gamma}_0 + \tilde{\gamma}_1)^2}{4} + \tilde{\alpha}_0\tilde{\beta}_1} = \frac{\tilde{\gamma}_0 + \tilde{\gamma}_1}{2} \pm \sqrt{\frac{(\tilde{\gamma}_0 + \tilde{\gamma}_1)^2}{4} + 24(A_1 + 4A_2)^2}. \end{aligned}$$

The conditions  $\alpha_L = 0$  and  $\tilde{\alpha}_M = 0$  are different, so in general the invariant subspace exists only for even or only for odd states. However, it may happen that both conditions are fulfilled. Thus, for  $L = 1 = M$  the compatibility of (10) and (13) demands  $A_1 = -5A_2$ ,  $A_0 = 6A_2$ . Then we have the simple explicit solutions for 4 levels of Hamiltonian (3):

$$E = \frac{\tilde{\gamma}_0 + \tilde{\gamma}_1}{2} \pm \sqrt{\frac{(\tilde{\gamma}_0 + \tilde{\gamma}_1)^2}{4} + 24A_2^2}, \quad \frac{\gamma_1}{2} \pm \sqrt{\frac{\gamma_1^2}{4} + 72A_2^2}.$$

#### IV. EXPLICIT SOLUTION FOR TWO LEVELS

If the coefficient  $\alpha_L = 0$ , the dimension of the invariant space is  $L + 1$ . Meanwhile, it may happen that, in addition,  $\beta_{L-1} = 0$ . Then the two-dimensional subspace spanned on  $\Phi_L$  and  $\Phi_{L-1}$  is singled out from the  $L + 1$  subspace that gives explicit simple exact solutions for two levels, whatever large  $L$  would be. For Hamiltonian (7) it follows from (8) that in this case

$$A_0 = 2L(2L - 3)A_2, \quad A_1 = A_2(3 - 4L), \quad (14)$$

$$\begin{aligned} \alpha_{L-1} &= -2A_2, \quad \beta_L = -4L(2L - 1)A_2, \\ E_{\pm} &= \frac{\gamma_L + \gamma_{L-1}}{2} \pm \sqrt{\frac{(\gamma_L - \gamma_{L-1})^2}{4} + \alpha_{L-1}\beta_L} = \frac{\gamma_L + \gamma_{L-1}}{2} \pm \sqrt{\frac{(\gamma_L - \gamma_{L-1})^2}{4} + 8L(2L - 1)A_2^2}. \end{aligned}$$

The similar procedure can be repeated for odd states.

#### V. GENERALIZATION

The obvious generalization of QES Bose Hamiltonians arises if Hamiltonian itself does not have a "canonical" structure under description but can be reduced to it with the help

of the transformation  $HK = KH'$ , where the operator  $K$  is some function of  $a$  and  $a^+$ . In particular, it can realize  $u - v$  Bogolubov transformations. In what follows we assume that such transformations, if needed, are already performed, so we try to generalize such "canonical" forms themselves. Consider the action of Hamiltonian  $H = H_0 + V$  with  $H_0$  from (3) and

$$V = \sum_{s=0}^{s_0} \sum_{k=1}^{k_0} A_{sk} [(a^+ a)^s a^{kq} + (a^+)^{kq} (a^+ a)^s] \quad (15)$$

on the functions  $\Phi_n = x^{qn}$ , where  $q > 0$  is an integer,  $n = 0, 1, \dots$ :

$$\begin{aligned} H_x \Phi_n &= \gamma_n \Phi_n + \sum_{k=1}^{k_0} \alpha_{nk} \Phi_{n+k} + \sum_{k=1}^{k_0} \beta_{nk} \Phi_{n-k}, \\ \gamma_n &= \sum_{p=1}^{p_0} \varepsilon_p (nq)^p, \\ \alpha_{nk} &= \sum_{s=0}^{s_0} A_{sk} (nq)^s, \\ \beta_{nk} &= \sum_{s=0}^{s_0} A_{sk} nq(nq-1)\dots(nq-kq+1)(nq-kq)^s, \quad n \geq k, \\ \beta_{nk} &= 0, \quad n < k. \end{aligned} \quad (16)$$

Provided the conditions of cut off at  $n = N$  are fulfilled, this Hamiltonian can possess the invariant subspace  $F = \{\Phi_n, n = 0, 1, \dots, N\}$ , so the wave function of the system  $\Phi = \sum_{n=0}^N a_n \Phi_n$  includes the states with  $nq$  ( $n = 0, 1, \dots, N$ ) particles only. The conditions under discussion read now as follows. For any given  $k$  we must demand  $\alpha_{N+1-i, k} = 0$ ,  $i = 1, \dots, k$ , so we have  $k$  conditions. As  $k = 1, 2, \dots, k_0$ , the total number of conditions is equal to  $n_1 = \frac{(k_0+1)k_0}{2}$ . On the other hand, the number of coefficients  $A_{sk}$  is equal to  $n_2 = (s_0+1)k_0$ . The system can be quasi-exactly solvable if  $n_2 > n_1$ , so  $2s_0 \geq k_0$ . In particular, in accordance with examples considered above, one can always adjust the coefficients properly, if  $k_0 = 1$ .

The approach considered in the present paper allows extension to many-particle systems. In particular, for two pairs of Bose operators  $(a, a^+)$ ,  $(b, b^+)$  one may take

$$H = \sum_i H_i^{(a)} h_i^{(b)}, \quad (17)$$

where Hamiltonians  $H_i$  and  $h_i$  are built from operators  $a, a^+$  and  $b, b^+$ , correspondingly, and have the structure (3) or (15).

## VI. CONCLUDING REMARKS

During recent years, the class of QES was extended considerably to include two- and many-dimensional systems, matrix models [9], the QES anharmonic oscillator with complex potentials ([6], p. 192; [10]), etc. Meanwhile, it turned out that, apart from these (sometimes rather sophisticated and exotic) situations, quasi-exact solvability exists in an everyday life around us where anharmonic Bose oscillators can be met at every step. In particular, the results obtained can be exploited in solid state or molecular physics, theory of magnetism, etc. The approach suggested in the present paper, shows the line along which a lot of second-quantized models with algebraization of the part of the spectrum can be constructed. This approach can be also extended to systems with interaction of subsystems of different nature - in particular, between spin and Bose operator, Bose and Fermi oscillators.

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